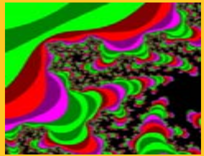

Extended-Precision Floating-Point Numbers for GPU Computation

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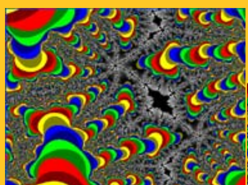
SUMMARY

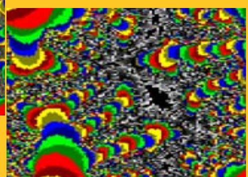
Using unevaluated sums of paired or quadrupled single-precision (f_{32}) values, *double-float* (df_{64}) and *quad-float* (qf_{128}) numeric types can be implemented on current GPUs and used efficiently and effectively for extended-precision computation for real and complex arithmetic. These numeric types provide 48 and 96 bits of precision respectively at f_{32} exponent ranges for computer graphics and general purpose (GPGPU) programming.

Rationale for and Prior Work on Extended Precision

Modern GPUs have wide data-buses allowing extremely high throughput, effecting a stream-computing model and allowing SIMD/MIMD computation at the fragment (pixel) level. Machine precision is limited, however, to 32-bit *nearly* IEEE 754 compliant floating-point. This limited precision of fragment-program computation presents a drawback for many GPGPU applications. GPU hardware will improve its IEEE compliance, but it is unlikely to support double or extended-precision arithmetic in the near future.

Techniques for performing extended-precision arithmetic in software using pairs of machine-precision numbers have a long history: [Dekker 1971], [Wyatt 1976], and [Brent 1978]; the *doubledoubles* of [Briggs 1998] and *quad-doubles* of [Hida et al 2001]. The use and hazards of these numerical types is discussed in [Li 2000].





The images in this example are areas of the Mandelbrot set in a region of the complex plane only **2.6×10^{-6}** across. The right image was computed on the GPU using f_{32} precision floats; the left image used complex cdf_{64} floats. The cdf_{64} code ran approximately 3x slower, averaging 8 frames/second for a 768x512 image; this includes time to compute an extended-precision square-root, unnecessary except to compare accuracy with d_{64} routines on the CPU.

Double-Float and Quad-Float Computation on GPUs

The hardware support for vector operations in fragment programs makes them well-suited for double- and quad-precision computation. Preliminary exploration of the feasibility and error-characteristics of GPU double-floats has been done using the Cg language [Meredith and Bremer 2004] and the Brook language [Da Graça and Defour 2006]. The current research was undertaken independently and uses Cg, as did Meredith's. We chose Cg, despite some drawbacks, in order to support other projects in interactive graphics, image-analysis and visualization.

This discussion will limit itself to df_{64} methods; techniques for qf_{128} generalize similarly from the methods described by Hida [5]. A df_{64} number A is an unevaluated sum of two f_{32} numbers represented in Cg as a 2-vector of single-precision floats:

$$A = a_{hi} + a_{lo}$$

`float2(a_hi, a_lo);`

Arithmetic operations on these numbers are done exactly using the a_{lo} term to give the error on the a_{hi} term. By allowing the a_{lo} term to be negative, we make use of the full 48-bits of precision available in the pair of values (23 bits + 1 hidden per mantissa).

```
float2 df64_add(float2 a, float2 b) {
   // use parallelism to simultaneously compute
   //   [(s, err), (t, err)] = [(a.x + b.x), (a.y, b.y)]
   float4 st = twoSumHiLo(a, b);
   st.y += st.z;
   st = quickTwoSum(st.x, st.y); // assumes |st.x| >= |st.y|
   st.y += st.w;
   return quickTwoSum(st.x, st.y);
}
```

```
float2 df64_mult(float2 a, float2 b) {
   // Multiply the hi-order terms at df64 precision,
   //   then add the low-order products
   float2 p = twoProd(a.x, b.x);
   p.y += dot(a, b.yx);
   return quickTwoSum(p.x, p.y); // renormalize product
}
```

```
float2 df64_exp(float2 a) {
   float2 outVal, r, rem, df_z;
   if (!specialCase(outVal)) {
       rem = df64_rem(a, df64M.LOG2, df_z);
       int z = (int) df_z;
       r = rem/64;
       r = df64_expTAYLOR(r);
       outVal = df64_mult(r, df64_npow(2.0*ONE, z));
   }
   return outVal;
}
```

Achievements of the Current Project

- Creating df_{64} data structures, basic arithmetic operations, and code for functions such as sqrt(x), ln(x), exp(x), and trigonometric functions taking df_{64} and f_{32} operands.
- Creating qf_{128} data structures and basic arithmetic operations and example functions;
- Creating a C++ **df₆₄** class for CPU-side interface with df_{64} fragment code, with tables of constants at df_{64} precision for π, e, ln(2).
- Creating test-applications to demonstrate the effectiveness and efficiency of extended-precision fragment code for graphics and GPGPU programming;
- Creating fast bitonic merging of addends for qf_{128}.
- Creating complex cdf_{64} numbers and demonstrating their efficiency.
- Creating test-applications to study error-generation and propagation by basic and library methods;

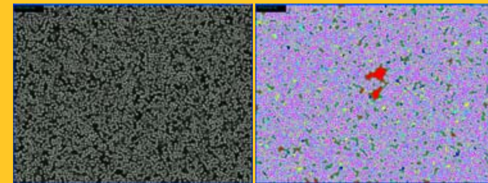
A sample routine showing the use of cdf_{64} complex variables to generate a the Mandelbrot set images used in the examples.

Improvements to current implementation:

- Full implementation of df_{64}, cdf_{64}, qf_{128} and cqf_{128} as first-order primitives in Cg using preprocessor or compiler modifications.
- More in-depth study of error behavior given the non-strict IEEE 754 compliance of current GPU hardware [Hillesland and Lastra 2004].
- Porting of these methods to Brook or CGiS or other non-graphics-oriented stream languages for GPGPU.
- Careful and correct generation of NaN and ±Inf values.
- Improvements to code for range-reduction in numerical routines.

Future work!

- GPU-based fractal- and non-fractal data compression.
- Porting of GPU FFT methods to df_{64} and qf_{128} precision routines for applications in image generation and analysis, and in computational number theory (e.g. Lucas-Lehmer testing for Mersenne primes).
- Application of df_{64} and qf_{128} to solving linear systems, PDEs, and physically-based graphics applications.
- Application to primitives to shading and illumination effects
- Modification of code to create extended precision integer class for numerical algorithms.



Solution of PDEs for simulation, such as the above reaction-diffusion textures, can benefit from extended precision: on the left, a Grey-Scott RD-texture (original code by Mark Harris); on the right, a similar simulation at df_{64} precision, showing relative concentrations of reactants.

DISCUSSION

Some points of interest:

1. Care must be taken to prevent compiler optimizations from altering or eliminated necessary computation. Defining **ONE** as a uniform parameter-multiplier for constants was only one of the necessary kludges to sidestep the aggressive **cgc** compiler.
2. GPUs have a MAD operator, equivalent to the numerical MAC (multiply-and-accumulate), allowing

   $$d = a*b + c$$

   to be evaluated in a single command; preliminary tests seem to indicate that this is not an FMAC (fused multiply-and-accumulate), which has a single roundoff error and would allow more efficient versions of extended-precision routines.
3. Because GPU operations can act on the four elements of a **float4** in a single operation, the above routines can be used for cdf_{64} complex values with no additional runtime overhead. In terms of these basic ops, we get complex numbers for free.

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